Some Recent Advancements in Monotone Circuit Complexity

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Abstract

In 1985, Razborov [16] proved the first superpolynomial size lower bound for monotone Boolean circuits for the perfect matching the clique functions, and, independently, Andreev [2] obtained exponential size lower bounds. These breakthroughs were soon followed by further advancements in monotone complexity, including better lower bounds for clique [1, 19], superlogarithmic depth lower bounds for connectivity by Karchmer and Wigderson [12], and the separations $\mathsf{mon\text{-}NC} \neq \mathsf{mon\text{-}P}$ and that $\mathsf{mon\text{-}NC}^i \neq \mathsf{mon\text{-}NC}^{i+1}$ by Raz and McKenzie [15]. Karchmer and Wigderson [12] proved their result by establishing a relation between communication complexity and (monotone) circuit depth, and Raz and McKenzie [15] introduced a new technique, now called *lifting theorems*, for obtaining communication lower bounds from query complexity lower bounds,

In this talk, we will survey recent advancements in monotone complexity driven by query-to-communication lifting theorems. A decade ago, Göös, Pitassi, and Watson [10] brought to light the generality of the result of Raz and McKenzie [15] and reignited this line of work. A notable extension is the lifting theorem [8] for a model of DAG-like communication [17, 18] that corresponds to circuit size. These powerful theorems, in their different flavours, have been instrumental in addressing many open questions in monotone circuit complexity, including: optimal $2^{\Omega(n)}$ lower bounds on the size of monotone Boolean formulas computing an explicit function in NP [14]; a complete picture of the relation between the $\mathsf{mon\text{-}AC}$ and $\mathsf{mon\text{-}NC}$ hierarchies [6]; a near optimal separation between monotone circuit and monotone formula size [5]; exponential separation between NC^2 and $\mathsf{mon\text{-}P}$ [8, 11]; and better lower bounds for clique [7, 13], improving on [3]. Very recently, lifting theorems were also used to prove supercritical trade-offs for monotone circuits showing that there are functions computable by small circuits for which any small circuit must have superlinear or even superpolynomial depth [4, 9]. We will explore these results and their implications, and conclude by discussing some open problems.

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Category Invited Talk

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